Partial synchronization of chaotic systems with uncertainty

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We suggest an approach to partial synchronization of chaotic systems with uncertainty. This method contains two steps: (i) transforming the synchronization system into the canonical form by the well-known feedback linearization theory and (ii) finding a control signal to ensure the asymptotic stability of the canonical system. This partial synchronization approach requires very little system information by applying a finite-time convergence technique to estimate uncertainties caused by unknown states, parameters, or structure. We also argue in detail that this partial synchronization method can be extended to parameter identification, (sub)structure estimation, and even phase detection. Several examples are presented to illustrate the partial synchronization approach suggested.

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I. INTRODUCTION

Synchronization [1,2] as a universal concept in nonlinear sciences has attracted much attention during the past years. Different types of synchronization have been observed, such as identical synchronization [3], phase synchronization [4], generalized synchronization [5–7], and lag synchronization [8], to name just a few; see Refs. [1,2] for a review. Identical synchronization is said to occur when states of two coupled identical (chaotic) systems (evolving from different initial conditions) coincide asymptotically. A perfect phase-locking phenomenon, called phase synchronization, can be observed for two weakly coupled systems with small parameter mismatch if their phases can be well defined. Generalized synchronization is said to happen when there exists a static functional relation between states of the driving and response systems if transients are ignored. Lag synchronization implies that two signals become identical in phases (and amplitudes), but shifted in time. These synchronization degrees can also be extended to networks of interacting dynamical systems [9]. Some authors also suggested to quantify the synchronization degree of interacting dynamical units in real biological systems and to reveal the relation between synchronization degree and functions of real biological systems; see, for example, Refs. [10–12].

Partial synchronization, on the other hand, occurs when only subsystems of both coupled systems are in synchrony while other state variables remain uncorrelated. Similarly, different types of partial synchronization can also be investigated, such as partial identical synchronization, partial phase synchronization, and partial generalized synchronization. Partial synchronization can be applied to describe the synchronization of two coupled oscillators with the same dimension or with different dimensions [13]. Partial synchronization can also be extended to networks of interacting dynamical systems [14–20].

Generally speaking, the research on synchronization may be grouped into two problems. The first one, which we call the analysis problem, consists of understanding and/or giving a theoretical description of synchronization phenomena. The second problem, the synthesis problem, is concerned with finding (or designing) a synchronization control signal such that coupled systems can synchronize with each other to some extent. Coupled complex dynamical systems in practice are nonidentical and have some uncertainty caused by unknown states, parameters, or structure. In this case, identical synchronization cannot easily be observed; whereas phase synchronization is not easy to be detected because the phase often cannot be well-defined or is not available for complex dynamical systems. For these reasons, the concepts of generalized synchronization or partial synchronization are often more practical to apply. However, until now some aspects especially important for applications are not well understood. Can we find some synchronization signal (or coupling schemes) to ensure generalized synchronization or partial synchronization of nonidentical coupled systems, especially when systems suffer from uncertainties caused by unknown states, parameters, or structure? Is it possible to find conditions under which a certain type of partial synchronization (e.g., partial identical synchronization, partial generalized synchronization, or partial phase synchronization) occurs? Can we estimate system parameters (including parameter mismatch), (sub)structure, or even phase from synchronization?

In this paper we suggest an approach to partial synchronization of chaotic systems with uncertainties caused by unknown states, parameters, or even structure based on a finite-time uncertainty estimation technique. Previously developed robust partial synchronization methods [13,21,22] achieved only bounded uncertainty estimation. Furthermore, we argue in detail that this partial synchronization method can be extended to parameter identification, (sub)structure estimation, or even phase detection [23].

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II. THEORY

A. Unidirectional coupling

We start with two unidirectionally coupled systems

$$\text{driving system: } \dot{\boldsymbol{x}}_d = \boldsymbol{f}_d(\boldsymbol{x}_d), \tag{1}$$

where $\boldsymbol{x}_d \in R^d$ is the state vector and $\boldsymbol{f}_d$ describes the system dynamics and

$$\text{response system: } \dot{\boldsymbol{x}}_r = \boldsymbol{f}_r(\boldsymbol{x}_r), \tag{2}$$

where $\boldsymbol{x}_r \in R^r$ denotes the state vector and $\boldsymbol{f}_r$ describes the system dynamics. We assume that both driving and response systems are chaotic and evolve from different initial conditions. This implies that without using a coupling signal, the response system quickly diverges from the trajectory of the driving system and no coherent linkage (or synchronization degree) between both systems can be observed.

Here we attempt to design a control signal u and vector $\boldsymbol{b}$ such that a controlled subsystem of the response system (2), given by

$$\dot{\boldsymbol{x}}_{r1} = \boldsymbol{f}_{r1}(\boldsymbol{x}_r) + \boldsymbol{b}u, \tag{3}$$

synchronizes with a subsystem of system (1), described by

$$\dot{\boldsymbol{x}}_{d1} = \boldsymbol{f}_{d1}(\boldsymbol{x}_d), \tag{4}$$

where $\boldsymbol{x}_{r1}, \boldsymbol{x}_{d1} \in \mathbb{R}^m$, $\boldsymbol{x}_r = (\boldsymbol{x}_{r1}, \boldsymbol{x}_{r2})$, $\boldsymbol{f}_r = (\boldsymbol{f}_{r1}, \boldsymbol{f}_{r2})$, $\boldsymbol{x}_d = (\boldsymbol{x}_{d1}, \boldsymbol{x}_{d2})$, and $\boldsymbol{f}_d = (\boldsymbol{f}_{d1}, \boldsymbol{f}_{d2})$. If we can perform this, partial synchronization between systems (1) and (2) is achieved.

It is easy to see that even when state vectors $\boldsymbol{x}_{d1}$ and $\boldsymbol{x}_{r1}$ are measured or observable, both subsystems (3) and (4) generally suffer from uncertainties caused by unknown states $\boldsymbol{x}_{r2}$ and $\boldsymbol{x}_{d2}$, parameters, or even structure. Therefore the essential issue of partial synchronization synthesis is associated with estimation of such uncertainties. In this paper we suggest a partial synchronization method with finite-time uncertainty estimation and show that under some rather general mathematical conditions, partial synchronization is ensured.

Let $\mathbf{e} = \boldsymbol{x}_{r1} - \boldsymbol{x}_{d1}$ and $\mathbf{e} = (e_1, \ldots, e_m)$. Then the synchronization error equation reads

$$\dot{\mathbf{e}} = \boldsymbol{f}_{r1}(\mathbf{e} + \boldsymbol{x}_{d1}, \boldsymbol{x}_{d2}) - \boldsymbol{f}_{d1}(\boldsymbol{x}_d) + \boldsymbol{b}u. \tag{5}$$

In the following, we assume that a scalar variable $z_1 = h(\mathbf{e})$ is measurable.

Let $z_{i+1} = \dot{z}_i$ for $i = 1, 2, \ldots, m-1$ and $\boldsymbol{z} = (z_1, z_2, \ldots, z_m)^T$. Then we obtain

$$\boldsymbol{z} = \mathcal{H}(\mathbf{e}, \boldsymbol{x}_d, u).$$

It follows from the well-known feedback-linearization theory [24] that if $\mathcal{H}$ does not contain u explicitly (i.e., $\partial\mathcal{H}/\partial u = \mathbf{0}$) and $\partial\mathcal{H}/\partial\mathbf{e}$ is nonsingular and continuous everywhere on a certain open set, then there exists a coordinate transformation

$$\boldsymbol{z} = \Phi(\mathbf{e}, \boldsymbol{x}_d) \tag{6}$$

such that the error system (5) can be globally transformed into the canonical form

$$\dot{z}_i = z_{i+1}, \quad i = 1, \ldots, m-1, \tag{7}$$
$$\dot{z}_m = \Delta(\boldsymbol{z}, \boldsymbol{x}_d) + u.$$

Here $\Delta(\boldsymbol{z}, \boldsymbol{x}_d)$ describes the dynamics of the transformed system. If the structure of coupled systems is known *a priori*, then $\Delta(\boldsymbol{z}, \boldsymbol{x}_d)$ can be calculated analytically. In this case, the asymptotic stability of system (7) can easily be done using well-developed control methods such as linear feedback control. However, the structure of coupled systems usually is not known exactly. Therefore, in practice some techniques have to be used for estimating $\Delta(\boldsymbol{z}, \boldsymbol{x}_d)$. As will be shown below, we shall apply a finite-time convergence strategy to estimate $\Delta(\boldsymbol{z}, \boldsymbol{x}_d)$ correctly within a finite time. In the following, we use Δ as an abbreviation of $\Delta(\boldsymbol{z}, \boldsymbol{x}_d)$ for simplicity.

Remark 1. Generally the coordinate transformation Φ in Eq. (6) can be grouped into three classes. The first class consists of transformations Φ that do not depend on $\boldsymbol{x}_d$ explicitly. The second one is that Φ contains $\boldsymbol{x}_{d1}$ explicitly but not $\boldsymbol{x}_{d2}$. The third class contains Φ depending on both $\boldsymbol{x}_{d1}$ and $\boldsymbol{x}_{d2}$ explicitly. If one can design a control signal u to ensure the asymptotic stability of system (7), then (i) partial identical synchronization between systems (1) and (2) occurs asymptotically for the first class of Φ because $\boldsymbol{z} \to \mathbf{0}$ implies $\mathbf{e} = \Phi^{-1}(\boldsymbol{z}) \to \mathbf{0}$; (ii) partial generalized synchronization is achieved asymptotically for the second class because $\boldsymbol{z} \to \mathbf{0}$ implies $\Phi(\boldsymbol{x}_{r1} - \boldsymbol{x}_{d1}, \boldsymbol{x}_{d1}) \to \mathbf{0}$; or (iii) more general (or weaker) partial generalized synchronization occurs asymptotically for the third class because $\boldsymbol{z} \to \mathbf{0}$ implies $\Phi(\boldsymbol{x}_{r1} - \boldsymbol{x}_{d1}, \boldsymbol{x}_d) \to \mathbf{0}$.

We now show how to design the control signal for ensuring the asymptotic stability of system (7). We first introduce a compositive error variable, described by

$$s = z_m + c_1 z_{m-1} + c_2 z_{m-2} + \cdots + c_{m-1} z_1, \tag{8}$$

where c_i's are designed such that polynomial $q^{m-1} + c_1 q^{m-2} + c_2 q^{m-3} + \cdots + c_{m-1}$ is Hurwitzian, which in combination with $z_{i+1} = \dot{z}_i$ ($\forall\ i = 1, 2, \ldots, m-1$) implies that $z_i \to 0$ for all i as $s \to 0$ and therefore the control goal considered becomes how to design the control signal to ensure $s \to 0$.

Differentiating Eq. (8) with respect to time and using Eq. (7), we then get

$$\dot{s} = \sum_{i=1}^{m-1} c_i z_{m-i+1} + \Delta + u. \tag{9}$$

The following theorems (see the Appendix for their proofs) summarize the rules to design the control signal to ensure $s \to 0$.

*Theorem 1***.** When Δ is known *a priori*, one can ensure $s = 0$ after a finite time by using the following control signal:

$$u = u_{\text{eq}} - k_1 s - k_2 s^p \operatorname{sgn}(s) - k_3 \int_0^t s\,dt - k_4 \left(\int_0^t s\,dt \right)^q \operatorname{sgn}\left(\int_0^t s\,dt \right), \tag{10}$$

where $u_{\text{eq}} = -\Delta - \sum_{i=1}^{m-1} c_i z_{m-i+1}$; k_1, k_2, k_3, and k_4 are positive constants; $0 < p, q < 1$; and $\operatorname{sgn}(\cdots)$ is the signum function.

Theorem 2. When Δ is unknown, provided that there exists a positive constant Γ such that $\dot{\Delta}(t)<\Gamma$ for all $t\geq 0$, one can ensure $s=0$ and

$$\Delta = \alpha|s-\hat{s}|^{1/2}\text{sgn}(s-\hat{s}) + \beta\int_0^t \text{sgn}(s-\hat{s})dt \quad (11)$$

after a finite time by using the following control signal:

$$u = -\sum_{i=1}^{m-1} c_i z_{m-i+1} + u_1, \quad (12)$$

where

$$u_1 = -k_1 s - k_2 s^p\,\text{sgn}(s) - k_3\int_0^t s dt - k_4\left(\int_0^t s dt\right)^q \text{sgn}\left(\int_0^t s dt\right) - \alpha|s-\hat{s}|^{1/2}\,\text{sgn}(s-\hat{s}) - \beta\int_0^t \text{sgn}(s-\hat{s})dt, \quad (13)$$

$$\hat{s} = -\int_0^t \left[k_1 s + k_2 s^p\,\text{sgn}(s) + k_3\int_0^t s dt + k_4\left(\int_0^t s dt\right)^q \text{sgn}\left(\int_0^t s dt\right)\right]dt, \quad (14)$$

with positive k_1, k_2, k_3, and k_4, $0<p,q<1$, $\alpha\geq 0.5\sqrt{\Gamma}$, and $\beta\geq 4\Gamma$. To summarize the above analysis, when the control signal u is designed according to theorem 1 or 2, one of the three types of partial synchronization as mentioned in remark 1 is ensured asymptotically, depending on the properties of Φ.

Remark 2. As shown in the proof of theorem 1 in Ref. [36], the time of ensuring Eq. (11) t^* can be estimated by

$$t^* \leq \frac{2}{\beta-\Gamma}\Omega(\alpha,\Gamma,\beta),$$

where $\Omega(\alpha,\Gamma,\beta)$ is a monotonically nonincreasing function of each of parameters α and β. Therefore, t^* can be shortened as desired if large enough parameters α and β are chosen.

Remark 3. As will be shown below, the assumption that there exists the coordinate transformation (6) such that the error system (5) can be globally transformed into the canonical form (7) is not really a restriction.

Remark 4. In most cases, Δ can be divided into two parts: one is called "certainty" which comes from the prior knowledge about coupled systems; the other "uncertainty." Theorem 1 is less significant because the "uncertainty" of Δ deteriorates the control performance dramatically. This drawback is removed by theorem 2 in which a finite-time estimation of uncertainty is involved, and can also be attacked by well-developed adaptive control techniques that enable estimating the "uncertainty" of Δ with any accuracy as desired. Indeed we may still adopt the control rule (10) but replace the true Δ by its "certainty" plus adaptive estimation value for the "uncertainty." For example, when Δ reads $\Delta(\boldsymbol{p}_1,\boldsymbol{p}_2)=\Sigma_i p_{1i}\phi_{1i}(\boldsymbol{x}_d)+\Sigma_i p_{2i}\phi_{2i}(z)$, the "uncertainty" of Δ arises from unknown parameters p_{1i} and p_{2i}. We can design an adaptive control signal [25] as

$$u = -\sum_{i=1}^{m} k_i e_i - k_1 s - k_2 s^p - k_3\int_0^t s dt - k_4\left(\int_0^t s dt\right)^q \text{sgn}\left(\int_0^t s dt\right) - \Delta(\boldsymbol{q}_1,\boldsymbol{q}_2),$$

$$\dot{q}_{1i} = \phi_{1i}(\boldsymbol{x}_d)z_m,$$

$\dot{q}_{2i}=\phi_{2i}(\boldsymbol{x}_d)z_m$.For a more general case when the structure of Δ is not precisely known, by taking $\phi_{1i}(\boldsymbol{x}_d)$ and $\phi_{2i}(\boldsymbol{x}_d)$ from kernel (or orthogonal basic) functions set (e.g., polynomial functions set), we can achieve increased accuracy if higher order kernel functions are contained in Δ.

Remark 5. If one system represents "reality" and the other a "computational model," the coupling signal can be used to achieve data assimilation [26–28] from a stream of noisy measurements into a running model that will effectively predict the (future) states of the true system. In data assimilation with an imperfect model, and more generally in a dynamic data driven applications system (DDDAS) [29], one seeks to change model parameters, as well as the model states, to match the true system more closely. In this case, Δ can be divided into two parts: one comes from "reality" and the other from the "computational model;" only the former is unknown and becomes uncertain whereas the latter is known exactly. In terms of theorem 2, this uncertainty part can be estimated after a finite time. Therefore, (partial) structure of "reality" can be identified. This can be extended to estimate system parameters and even to estimate system phase, as will be illustrated below (see Sec. III).

B. Bidirectional coupling

We now show that the control method used for unidirectional coupling can also be extended to bidirectionally coupled systems. For two systems (1) and (2), we assume that only the first state variables of them are coupled to each other. This implies that coupled subsystems can be described by

$$\dot{x}_{d1} = f_{d1}(\boldsymbol{x}_d) + v_1, \quad (15)$$

$$\dot{x}_{r1} = f_{r1}(\boldsymbol{x}_r) + v_2, \quad (16)$$

where $\boldsymbol{x}_d=(x_{d1},\boldsymbol{x}_{d2})$, $\boldsymbol{x}_r=(x_{r1},\boldsymbol{x}_{r2})$, $\boldsymbol{f}_d=(f_{d1},\boldsymbol{f}_{d2})$, $\boldsymbol{f}_r=(f_{r1},\boldsymbol{f}_{r2})$, and v_1 and v_2 are coupling signals to be specified. For simplicity, we set $v_1=-v/2$ and $v_2=v/2$.

Let $e=x_{r1}-x_{d1}$. Then the synchronization error equation reads

$$\dot{e} = \Delta + v. \quad (17)$$

Here $\Delta=f_{r1}(\boldsymbol{x}_r)-f_{d1}(\boldsymbol{x}_d)$ describes the difference between the two subsystems (15) and (16).

In this case, we design $s=e$ and Eq. (9) actually reads

$$\dot{s} = \Delta + v.$$

In terms of theorem 2, we can ensure $e(=s)=0$ after a finite time by designing the control signal v as

$$v(t) = -k_1 e - k_2 e^p \operatorname{sgn}(e) - k_3 \int_0^t e dt - k_4 \left(\int_0^t e dt \right)^q \operatorname{sgn}\left(\int_0^t e dt \right) - \alpha |e - \hat{e}|^{1/2} \operatorname{sgn}(e - \hat{e}) - \beta \int_0^t \operatorname{sgn}(e - \hat{e}) dt, \quad (18)$$

with

$$\hat{e} = -\int_0^t \left[k_1 e + k_2 e^p \operatorname{sgn}(e) + k_3 \int_0^t e dt + k_4 \left(\int_0^t e dt \right)^q \operatorname{sgn}\left(\int_0^t e dt \right) \right] dt. \quad (19)$$

It follows that partial identical synchronization between systems (1) and (2) occurs after a finite time. Furthermore, Δ can be estimated after a finite time by the following equation:

$$\Delta = \alpha |e - \hat{e}|^{1/2} \operatorname{sgn}(e - \hat{e}) + \beta \int_0^t \operatorname{sgn}(e - \hat{e}) dt. \quad (20)$$

Remark 6. This partial synchronization approach can be extended to a more general case when arbitrary state variables of systems (1) and (2) are coupled to each other. Because coupling signals v_i include dynamical elements, i.e., integral operators, such coupling signals are called dynamical coupling signals. Similarly, coupling signals that do not include dynamical elements are called static coupling signals. It should be noted that due to the presence of uncertainty Δ, one cannot ensure synchronization using the widely used static coupling form $v_1=-k_1(x_{d1}-x_{r1})$ and $v_2=-k_2(x_{r1}-x_{d1})$, where k_1 and k_2 are control gains. However, (partial) synchronization may occur with proper dynamical coupling signals.

III. EXAMPLES

We now present some examples to illustrate the partial synchronization method suggested.

Example 1. As the first example, we consider a chaotic circuit with a double-scroll attractor [30]

$$\dot{x}_{1d} = x_{2d}, \quad \dot{x}_{2d} = x_{3d}, \quad \dot{x}_{3d} = -\alpha[x_{1d} - \operatorname{sgn}(x_{1d})] + x_{2d} + x_{3d} \quad (21)$$

as the "driving" system, and a Duffing model [31,32]

$$\dot{x}_{1r} = x_{2r}, \quad (22)$$

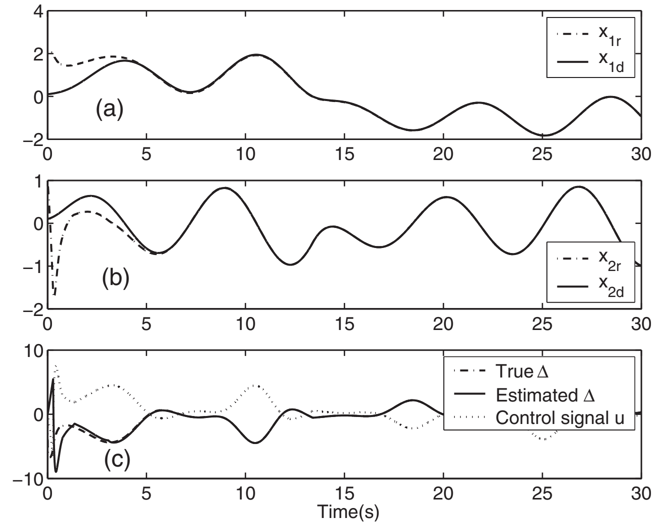

FIG. 1. Partial identical synchronization between systems with different dimension. (a) x_{1r} and x_{1d} versus time; (b) x_{2r} and x_{2d} versus time. (c) True Δ (dashed-dotted line) and its estimation (11) (solid line) versus time; and control signal u (dotted line) versus time.

$$\dot{x}_{2r} = -p_1 x_{1r} - p_2 x_{2r} - p_3 x_{1r}^3 + p_4 \cos(w_r t) + p_5 \sin(w_r t) + u$$

as the "response" system, where the parameters used in the following numerical simulations were $\alpha=0.8$, $p_1=1$, $p_2=0.25$, $p_3=1$, $p_4=0.3$, $p_5=0$, and $w_r=1$. The goal considered here is to design the control signal u such that system (22) synchronizes with the subsystem of the "driving" system, given by

$$\dot{x}_{1d} = x_{2d}, \quad \dot{x}_{2d} = x_{3d}. \quad (23)$$

Let $e_1=x_{1r}-x_{1d}$ and $e_2=x_{2r}-x_{2d}$. Then the synchronization error equation reads

$$\dot{e}_1 = e_2, \quad \dot{e}_2 = -x_{3d} - p_1 x_{1r} - p_2 x_{2r} - p_3 x_{1r}^3 + p_4 \cos(w_r t) + p_5 \sin(w_r t) + u. \quad (24)$$

Let $z_1=e_1$ and $z_2=e_2$. Then the error system (24) can be globally transformed into the canonical form

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = \Delta + u, \quad (25)$$

where

$$\Delta = -x_{3d} - p_1 x_{1r} - p_2 x_{2r} - p_3 x_{1r}^3 + p_4 \cos(w_r t) + p_5 \sin(w_r t).$$

It is easy to see that Eq. (25) has the same form as in Eq. (7). According to theorem 2, we can design the control signal u as in Eq. (12) with $s=z_2+z_1$, $k_1=k_2=k_3=k_4=1$, $p=q=3/5$, and $\alpha=\beta=20$, such that $z_1,z_2\to 0$ (therefore $x_{1r}\to x_{1d}$ and $x_{2r}\to x_{2d}$), and in addition, the uncertainty Δ can be estimated using Eq. (11) after a finite time. This is illustrated by simulations shown in Fig. 1. It is easy to see from Figs. 1(a) and 1(b) that partial identical synchronization between systems (21) and (22) occurs after a short time. This is consistent with the fact that Φ belongs to the first class and does not contain $\boldsymbol{x}_d$ explicitly.

Figure 1(c) shows that the uncertainty Δ can be estimated using Eq. (11) after a short time. It should be stressed that this method does not depend on the detailed information of the third equation of system (21). In fact it is applicable to a more general system as the driving system, given by

$$\dot{x}_{1d}=x_{2d},\quad \dot{x}_{2d}=x_{3d},\quad \dot{x}_{3d}=\chi(x_{1d},x_{2d},x_{3d}), \tag{26}$$

where χ is freely chosen.

Noting that the term $-x_{3d}-p_1x_{1d}-p_2x_{2d}-p_3x_{1d}^3+p_4\cos(w_rt)+p_5\sin(w_rt)$ can be estimated by Δ because of $x_{2r}\to x_{2d}$ and $x_{1r}\to x_{1d}$, we now show that this uncertainty estimation method in combination with parametric fitting methods can be extended to identify all parameters of system (22), namely, p_i and w_r, if proper x_{1d}, x_{2d}, and x_{3d} are designed for system (26) and all state variables of the driving system (26) are measurable.

First, we choose $x_{1d}(t)\equiv 1$ and $x_{2d}(t)\equiv x_{3d}(t)\equiv 0$ for all $t\geq 0$. Similarly, as shown in Fig. 1(c), when $t>t^*$, $\Delta(t)=-p_1-p_3+p_4\cos(w_rt)+p_5\sin(w_rt)$ is ensured because of $x_{2r}\to 0$ and $x_{1r}\to 1$. We assume that when $t>t^*$, Δ has the form

$$\Delta(t,a_0,a_1,b_1,w_0)=a_0+a_1\cos(w_0t)+b_1\sin(w_0t), \tag{27}$$

where $a_0\triangleq -p_1-p_3$, $a_1\triangleq p_4$, $b_1\triangleq p_5$, and $w_0\triangleq w_r$. By sampling $\Delta(t)$ with proper rate S_r from the interval $[t_s,t_f]$ $(t_s>t^*)$, we can construct a data pairs set $\{(t_k,\Delta_k)\}$ for parametric fitting and then attempt to find parameters $\bar{a}_0$, $\bar{a}_1$, $\bar{b}_1$, and $\bar{w}_0$ such that the root-mean-square error (RMSE), given by

$$\sqrt{\frac{1}{N}\sum_{k=1}^{N}[\Delta(t_k,a_0,a_1,b_1,w_0)-\Delta(t_k,\bar{a}_0,\bar{a}_1,\bar{b}_1,\bar{w}_0)]^2},$$

is minimal. In the following simulations, we set $t_s=40$ s, $t_f=60$ s, $S_r=2\times 10^{-4}$ s. By parametric fitting methods for Fourier series, we estimated parameters as $a_0=1.997$, $a_1=0.2999$, $b_1=1.771\times 10^{-4}$, and $w_0=1$, for which the RMSE is equal to 8.913×10^{-4}. Figure 2(a) shows the fitting results. Therefore $p_4=0.2999$, $p_5=1.771\times 10^{-4}$, $w_r=1$, and $p_1+p_3=1.997$ are estimated for their true values.

Second, we choose $x_{1d}(t)\equiv 0.5$ and $x_{2d}(t)\equiv x_{3d}(t)\equiv 0$ for all $t\geq 0$ such that when $t>t^*$, $\Delta(t)=-0.8p_1-0.8^3p_3+p_4\cos(w_rt)+p_5\sin(w_rt)$ is ensured because of $x_{2r}\to 0$ and $x_{1r}\to 0.8$. Similarly, we assume that Δ has the same form as in Eq. (27) and that we can estimate true $a_0=-0.8p_1-0.8^3p_3$ by parametric fitting methods for Fourier series. Figure 2(b) shows the fitting results, in which $t_s=40$ s, $t_f=60$ s, and $S_r=2\times 10^{-4}$ s were used and the RMSE is equal to 6.455×10^{-4}. Then a_0 was estimated as $a_0=-0.8p_1-0.8^3p_3=-1.31$, which in combination with $p_1+p_3=1.997$ leads to $p_1=0.9984$ and $p_3=0.9986$. Now only parameter p_2 is left to be further estimated (other parameters have been identified with high accuracy).

Finally, we choose $x_{1d}(t)=\sin(t)$, $x_{2d}(t)=\cos(t)$, and $x_{3d}=-\sin(t)$ for all $t\geq 0$ such that when $t>t^*$, $\Delta(t)=\sin(t)-p_1\sin(t)-p_2\cos(t)-p_3\sin^3(t)+p_4\cos(w_rt)+p_5\sin(w_rt)$ is ensured because of $x_{2r}\to\cos(t)$ and $x_{1r}\to\sin(t)$. Consequently $p_2\cos(t)=-\Delta(t)+\sin(t)-p_1\sin(t)-p_3\sin^3(t)+p_4\cos(w_rt)+p_5\sin(w_rt)$ can also be estimated, where we used the estimated values for p_1, p_3, p_4, p_5, and w_r. We set $t_s=40$ s, $t_f=60$ s, and $S_r=2\times 10^{-4}$ s. By fitting methods for Fourier series, p_2 can be estimated as $p_2=0.2501$, for which the RMSE is equal to -1.005×10^{-5}. Figure 2(c) shows the fitting results.

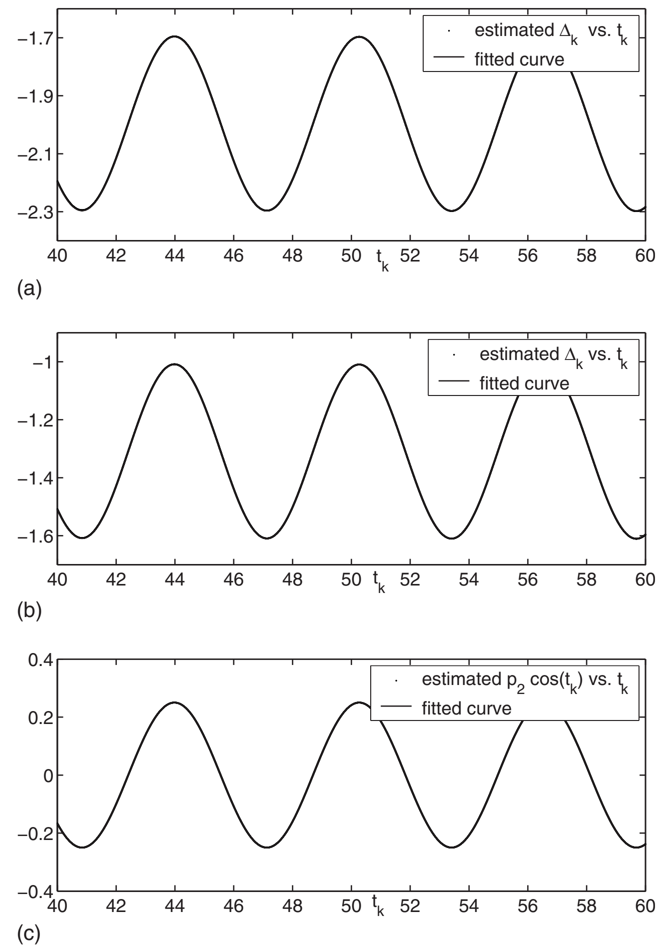

FIG. 2. Parametric fitting results. (a) Estimated $\Delta(t_k)$ (dot) versus t_k and fitted curve (solid line) when $x_{1d}(t)\equiv 1$ and $x_{2d}(t)\equiv x_{3d}(t)\equiv 0$. (b) Estimated $\Delta(t_k)$ (dot) versus t_k and fitted curve (solid line) when $x_{1d}(t)\equiv 0.8$ and $x_{2d}(t)\equiv x_{3d}(t)\equiv 0$. (c) Estimated $p_2\cos(t_k)$ (dot) versus t_k and fitted curve (solid line) when $x_{1d}(t)=\sin(t)$, $x_{2d}(t)=\cos(t)$, and $x_{3d}(t)=-\sin(t)$.

To summarize the above analysis, all parameters of system (22) have been estimated as $p_1=0.9984$, $p_2=0.2501$, $p_3=0.9986$, $p_4=0.2999$, $p_5=1.771\times 10^{-4}$, and $w_r=1$. Following the same steps as for the Duffing system, this parameter estimation method can be extended to more general systems described by

$$\dot{x}_{1r}=x_{2r},$$

$$\dot{x}_{2r}=\sum_k p_{1k}x_{1r}^k+\sum_k p_{2k}x_{2r}^k+\sum_k p_{3k}\cos(w_kt)+p_{4k}\sin(w_kt).$$

Example 2. Next, we consider partial synchronization of two systems with parameter mismatch. As an illustrating example, we analyze the Rössler systems [33] with parameter mismatch, given by

$$\dot{x}_{1,2} = -w_{1,2}y_{1,2} - z_{1,2}, \quad \dot{y}_{1,2} = w_{1,2}x_{1,2} + a_{1,2}y_{1,2}, \quad (28)$$

$$\dot{z}_{1,2} = b_{1,2} + z_{1,2}(x_{1,2} - c_{1,2}),$$

where parameters used in the following simulations were $w_1=0.985$, $a_1=0.15$, $b_1=0.2$, $c_1=10$, $w_2=1.015$, $a_2=0.1$, $b_2=0.25$, and $c_2=8.5$. We apply the control signal u to the second equation of the second system such that both systems synchronize with each other in the sense of partial synchronization.

Let $e=y_2-y_1$. Then the synchronization error equation reads

$$\dot{e} = \Delta + u, \quad (29)$$

where $\Delta=w_2x_2-w_1x_1+a_2y_2-a_1y_1$. In this case, we design $s=e$ and Eq. (9) actually reads

$$\dot{s} = \Delta + u.$$

It follows that the control signal (12) actually reads

$$u(t) = -k_1e - k_2e^p\,\mathrm{sgn}(e) - k_3\int_0^t edt - k_4\left(\int_0^t edt\right)^q \mathrm{sgn}\left(\int_0^t edt\right) - \alpha|e-\hat{e}|^{1/2}\,\mathrm{sgn}(e-\hat{e}) - \beta\int_0^t \mathrm{sgn}(e-\hat{e})dt, \quad (30)$$

with

$$\hat{e} = -\int_0^t\left[k_1e + k_2e^p\,\mathrm{sgn}(e) + k_3\int_0^t edt + k_4\left(\int_0^t edt\right)^q \mathrm{sgn}\left(\int_0^t edt\right)\right]dt. \quad (31)$$

According to theorem 2, we can ensure $e(=s)=0$ after a finite time by using the control signal (30). This implies that partial identical synchronization between the two Rössler systems occurs after a finite time. Furthermore, Δ can be estimated by the following equation:

$$\Delta = \alpha|e-\hat{e}|^{1/2}\,\mathrm{sgn}(e-\hat{e}) + \beta\int_0^t \mathrm{sgn}(e-\hat{e})dt. \quad (32)$$

Figures 3 and 4 summarize our results. It is easy to see from Figs. 3(a)–3(c) that partial identical synchronization between the two Rössler systems is ensured after a short time. Figure 3(e) shows that Δ can be estimated using Eq. (32). Noting $\Delta=w_2x_2-w_1x_1+a_2y_2-a_1y_1$, it follows that true $w_1x_1=-\Delta+w_2x_2+a_2y_2-a_1y_1$ can also be estimated after a short time, see Fig. 3(f).

We now show that the estimated w_1x_1 can be applied to phase evaluation. Indeed, it is well known that when $w_1=0.985$, $a_1=0.15$, $b_1=0.2$, $c_1=10$, the trajectory in the coordinates (x_1,y_1) rotates around the origin and therefore the phase of the driving Rössler system is well-defined as $\phi_1=\arctan(y_1/x_1)$, see Fig. 4(a). When $w_1\approx 1$, the true phase of the driving Rössler system, ϕ_1, can be estimated by

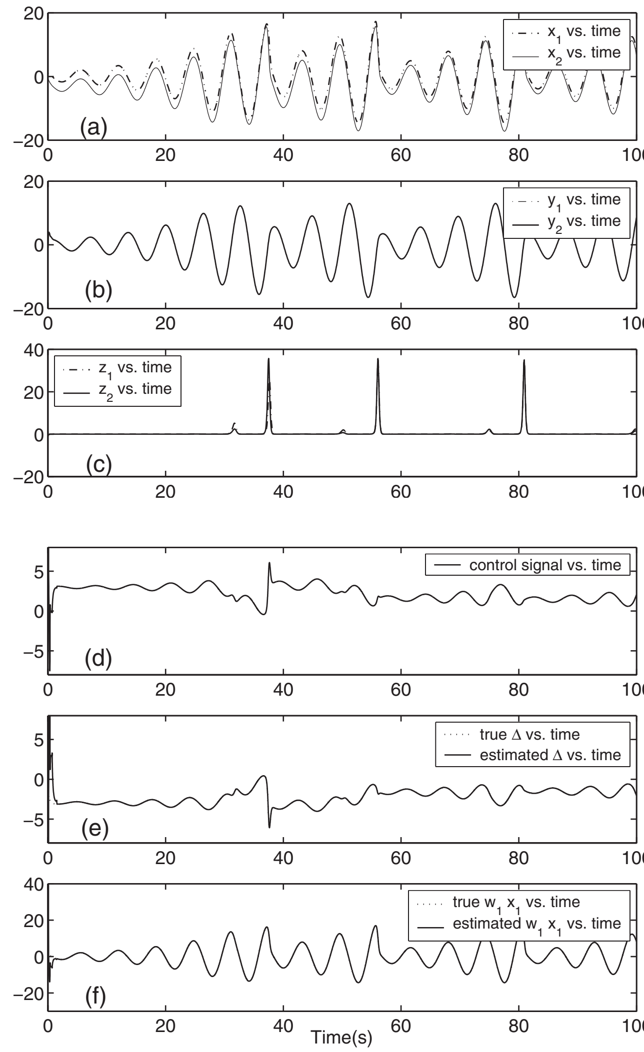

FIG. 3. Partial identical synchronization of two nonidentical Rössler systems. (a)–(c) States of two Rössler systems Rössler systems versus time. (d) Control signal versus time. (e) True Δ and its estimation (32) versus time. (f) True w_1x_1 and its estimation versus time.

$\arctan[y_1/(w_1x_1)]$, where the estimated value for w_1x_1 was used. Even when $w_1\approx 1$ is not satisfied, the average circle frequency of the driving Rössler system can be estimated.

We found that the trajectory in the coordinates (x_2,y_2) rotates around $(-2,0)$ and therefore the phase of the response Rössler system is well defined as $\phi_2=\arctan[y_2/(x_2+2)]$, see Fig. 4(b). Furthermore phase synchronization between the two Rössler systems occurs, which is shown in Fig. 4(c). This indicates that partial synchronization implies phase synchronization for some cases and therefore the phase of the driving system can be estimated from that of the response system (that is, the response system can be considered as a phase detector [23] to estimate the phase of the driving system). For example, as illustrated in Fig. 4(c), the true phase of the driving Rössler system ϕ_1 can be estimated by that of the response Rössler system ϕ_2. If $y_1(t)$ is obtained from the real Rössler system, the response Rössler system can then be considered as a phase detector to estimate the phase of real Rössler system. We are now extending this phase estimation

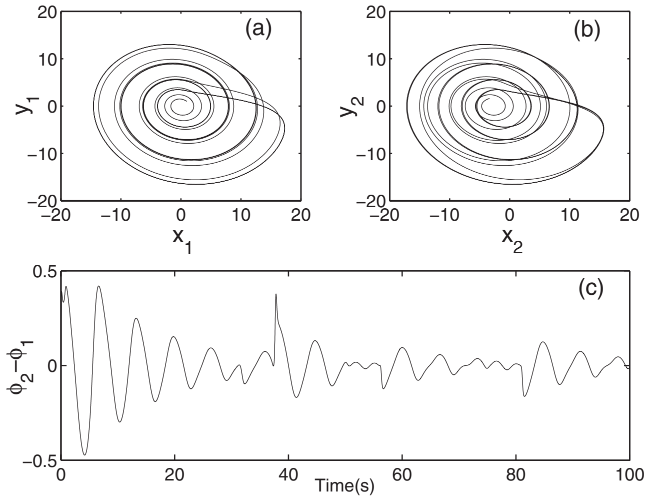

FIG. 4. Phase estimation using partial synchronization. (a) y_1 versus x_1. (b) y_2 versus x_2. (c) Phase difference $\phi_2-\phi_1$ versus time.

method to more general systems and are investigating to find conditions under which partial synchronization implies phase synchronization.

Example 3. Finally, we treat partial synchronization of structurally different systems by bidirectional (or mutual) couplings. As an example, we consider the Lorenz system [34] and the chaotic circuit (21) and assume that only the first state variables of both systems are coupled to each other. The coupled system is then described as

$$\dot{x}_{1d} = \sigma(x_{2d} - x_{1d}) + v_1, \quad \dot{x}_{2d} = \rho x_{1d} - x_{2d} - x_{1d}x_{3d}, \quad (33)$$

$$\dot{x}_{3d} = x_{1d}x_{2d} - bx_{3d}$$

and

$$\dot{x}_{1r} = x_{2r} + v_2, \quad \dot{x}_{2r} = x_{3r}, \quad (34)$$

$$\dot{x}_{3r} = -\alpha[x_{1r} - \operatorname{sgn}(x_{1r})] + x_{2r} + x_{3r},$$

where v_1 and v_2 are coupling signals to be designed. Again, we set $v_1 = -v/2$ and $v_2 = v/2$ and design v as in Eq. (18) such that $e = x_{1r} - x_{1d} = 0$ is ensured after a finite time. Figure 5 shows that partial identical synchronization between systems (33) and (34) occurs after a short time.

IV. CONCLUSIONS

Partial synchronization synthesis problems are systematically investigated for both unidirectionally and bidirectionally coupled systems. They are associated with the estimation of uncertainties caused by the unknown states, parameters, or structure. We suggest a robust method to partial synchronization with a finite-time uncertainty estimation technique. Furthermore we show in detail that this partial synchronization method can be extended to parameter identification and even phase detection. Although we have illustrated that partial synchronization of two bidirectionally coupled systems with different structure can be achieved by applying dynamical couplings, it is still an open problem how to extend this dynamical coupling method to complex networks of interacting (identical or nonidentical) dynamical systems. Some further research in this line is now under our investigation.

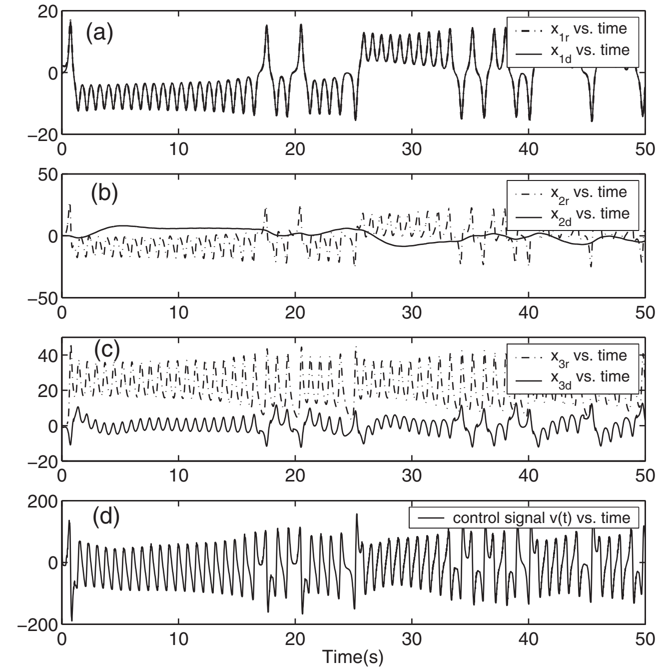

FIG. 5. Partial identical synchronization between bidirectionally coupled systems with different structure. (a) States x_{1d} and x_{1r} versus time. (b) States x_{2d} and x_{2r} versus time. (c) States x_{3d} and x_{3r} versus time. (d) Control signal versus time.

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APPENDIX

The following two lemmas will be applied in proofs below.

Lemma 1 [35]. Let $\boldsymbol{x} \in \boldsymbol{D} \subset \mathbb{R}^n$, $\dot{\boldsymbol{x}} \in \boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^n$ is continuous on an open neighborhood $\boldsymbol{D}$ of the origin and locally Lipshitz on $\boldsymbol{D}\backslash\{0\}$, and $\boldsymbol{f}(0)=0$. Suppose there exists a continuous function $V: \boldsymbol{D} \to \mathbb{R}$ such that (i) V is positive definite; (ii) $\dot{V}$ is negative on $\boldsymbol{D}\backslash\{0\}$; and (iii) there exist real numbers $k>0$ and $\alpha \in (0,1)$, and a neighborhood $N \subset \boldsymbol{D}$ of the origin such that $\dot{V} + kV^{\alpha} \leq 0$ on $N\backslash\{0\}$. Then the origin is a finite-time-stable equilibrium of $\dot{\boldsymbol{x}} \in \boldsymbol{f}(\boldsymbol{x})$.

Lemma 2 [36]. Consider the equation

$$\ddot{\sigma} + \frac{1}{2}\alpha|\sigma|^{-1/2}\operatorname{sgn}(\sigma) + \beta\int_0^t \operatorname{sgn}(\sigma)dt = \varsigma(t)$$

where $|\varsigma(t)| < C$ is continuous almost everywhere. Then $\dot{\sigma} = \sigma = 0$ is satisfied after a finite time if $\alpha \geq 0.5\sqrt{C}$ and $\beta \geq 4C$.

1. Proof of theorem 1

Substituting Eq. (10) into Eq. (9) yields

$$\dot{s} = -k_1 s - k_2 s^p \operatorname{sgn}(s) - k_3 \int_0^t s dt - k_4 \left(\int_0^t s dt\right)^q \operatorname{sgn}\left(\int_0^t s dt\right). \tag{A1}$$

Let $\xi_1 \equiv \int_0^t s dt$ and $\xi_2 \equiv s$. Then Eq. (A1) can be rewritten as

$$\dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = -k_1 \xi_2 - k_2 \xi_2^p \operatorname{sgn}(\xi_2) - k_3 \xi_1 - k_4 \xi_1^q \operatorname{sgn}(\xi_1). \tag{A2}$$

For system (A2), we choose a Lyapunov function

$$V(\xi_1,\xi_2) = k_3 \xi_1^2 + \frac{k_4}{q+1}|\xi_1|^{q+1} + \frac{1}{2}\xi_2^2 \tag{A3}$$

which is positive definite and thus fulfills condition (i) of lemma 1. Differentiating Eq. (A3) with respect to time yields

$$\dot{V} = -k_1 \xi_2^2 - k_2 \xi_2^{1+p}. \tag{A4}$$

This leads to that condition (ii) of Lemma 1 is guaranteed because the origin is the only equilibrium point of system (A2).

We can conclude from Eqs. (A3) and (A4) that

$$\dot{V} + \rho V^\epsilon = -k_1 \xi_2^2 - k_2 \xi_2^{1+p} + \rho\left(k_3 \xi_1^2 + \frac{k_4}{q+1}|\xi_1|^{q+1} + \frac{1}{2}\xi_2^2\right)^\epsilon,$$

where ρ is positive and ϵ is chosen such that $2\epsilon > 1+\rho$. Let $N \equiv \{(\xi_1,\xi_2) | \xi_1 \to 0, \rho(\frac{1}{2}\xi_2^2)^\epsilon < k_1\xi_2^2 + k_2\xi_2^{1+p}\}$. It is easy to see that due to $2\epsilon > 1+\rho$, $\rho(\frac{1}{2}\xi_2^2)^\epsilon < k_1\xi_2^2 + k_2\xi_2^{1+p}$ is satisfied when $|\xi_2| \ll 1$. When $(\xi_1,\xi_2) \in N$, $\rho(\frac{1}{2}\xi_2^2)^\epsilon$ domains $\rho(k_3\xi_1^2 + \frac{k_4}{q+1}|\xi_1|^{q+1} + \frac{1}{2}\xi_2^2)^\epsilon$ and thereby $\dot{V} + \rho V^\epsilon < 0$ is satisfied. It follows that condition (iii) of lemma 1 is also satisfied. According to lemma 1, the origin of system (A2) is finite-time stable. This theorem is thereby proved.

2. Proof of theorem 2

Substituting Eq. (12) into Eq. (9) yields

$$\dot{s} = \Delta + u_1. \tag{A5}$$

On the other hand, differentiating Eq. (14) with respect to time and using Eq. (13), we get

$$\dot{\hat{s}} = u_1 + \alpha|s - \hat{s}|^{1/2} \operatorname{sgn}(s-\hat{s}) + \beta \int_0^t \operatorname{sgn}(s-\hat{s}) dt. \tag{A6}$$

Let $\sigma = s - \hat{s}$. Subtracting Eq. (A6) from Eq. (A5) then results in

$$\dot{\sigma} = \Delta - \alpha|\sigma|^{1/2} \operatorname{sgn}(\sigma) - \beta \int_0^t \operatorname{sgn}(\sigma) dt. \tag{A7}$$

Differentiating Eq. (A7) with respect to time indicates

$$\ddot{\sigma} = \dot{\Delta} - \frac{1}{2}\alpha\dot{\sigma}|\sigma|^{-1/2} - \beta \operatorname{sgn}(\sigma).$$

According to lemma 2, it follows that $\dot{\sigma}=0$ is ensured after a finite time t^* and thereby when $t > t^*$, Eq. (A7) actually becomes

$$\Delta = \alpha|\sigma|^{1/2} \operatorname{sgn}(\sigma) + \beta \int_0^t \operatorname{sgn}(\sigma) dt,$$

which implies that Eq. (11) is satisfied after a finite time.

Substituted in Eq. (13) this implies that when $t > t^*$, Eq. (13) actually reads

$$u_1 = -k_1 s - k_2 s^p \operatorname{sgn}(s) - k_3 \int_0^t s dt - k_4 \left(\int_0^t s dt\right)^q \operatorname{sgn}\left(\int_0^t s dt\right) - \Delta,$$

and then

$$u = -\sum_{i=1}^{m-1} c_i e_{m-i+1} - \Delta - k_1 s - k_2 s^p \operatorname{sgn}(s) - k_3 \int_0^t s dt - k_4 \left(\int_0^t s dt\right)^q \operatorname{sgn}\left(\int_0^t s dt\right). \tag{A8}$$

Clearly Eq. (A8) has the same form as in Eq. (10). By theorem 1 this theorem is thereby proved.

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